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# Transcendents defined by nonlinear fourth-order ordinary differential equations

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**Abstract.** Three ordinary differential equations are considered. The general solutions of these equations are shown to be the essentially transcendental functions with respect to their initial conditions. Irreducibility of these equations is discussed.

## 1. Introduction

The problem of defining new functions by means of nonlinear ordinary differential equations (ODEs) was stated by Fuchs and Poincare. These ODEs must possess two important properties: irreducibility and uniformization of their solutions. The first property means that there exists no transformation, again within a precise class, reducing any of these equations either to a linear equation or to another order equation [1, 2]. The second property corresponds to the Painlevé property of an ODE because the absence of movable critical singularities in its general solution leads to the single-valued function.

Almost a century ago Painlevé and his school began a study the second-order ODE class [3]. They had two related objectives: to classify the second-order equations of a certain form on the basis of their possible singularities, and to identity equations of second order that essentially define new functions. Painlevé and his collaborators showed that out of all possible equations of a certain form, there are only 50 types which have the property of having no movable critical points. Furthermore, they showed that of these 50 equations, 44 were integrable in terms of previously known functions (such as elliptic functions and linear equations) or were reducible to one of six new nonlinear ODEs. They also showed that there are exactly six second-order equations that define new functions [3]. The functions defined by them are now called the six Painlevé transcendents. Later, Bureau extended Painlevé's first objective, and gave a partial classification of third-order equations [4–6]. The results of Painlevé and his collaborators led to the problem of finding other new functions that could be defined by nonlinear ODEs like the Painlevé transcendents. However, despite huge efforts, no new function has yet been found. In fact, no irreducible equation has been discovered since 1906 [1].

Although the six Painlevé equations were first discovered from strictly mathematical considerations, they have recently appeared in several physical applications [3].

Current interest in the Painlevé property is known to stem from the observations made by Ablowitz and Segur [7] and Ablowitz *et al* [8, 9] that reductions of partial differential equations of the soliton type give rise to ODEs whose movable singularities are only poles. This circumstance reduced them to the famous Painlevé conjecture, the Painlevé ODE test

[3]. 'Every ordinary differential equation which arises as a similarity reduction of a complete integrable partial differential equation is of Painlevé type, perhaps after a transformation of variables'. The Painlevé ODE test is applied as follows: if a given partial differential equation reducible to an ODE is not of Painlevé type then the Painlevé ODE test predicts that the partial differential equation is not complete integrable [3]. This test allows us to also find new ODEs of Painlevé type if we have the nonlinear integrable partial differential equation. We will use this in this work.

The aim of this paper is to show that the general solutions of three ODEs of fourth order, introduced as reductions of the integrable partial differential equations, are essentially transcendental functions with respect to their initial conditions.

The outline of this work is as follows. Three ODEs are presented in section 2. The approach that we use at the proof, namely that the general solutions of equations studied are the transcendental functions with respect to their constants of integration, is discussed in section 3. The proofs that the general solutions of the three equations are transcendental functions with respect to constants of integration are given in sections 4–6. Irreducibility of studied equations is discussed in section 7.

## 2. Equations studied

In a recent work [10] we presented a hierarchy which takes the form

$$d^{n+1}(u) = \tfrac{1}{2}z \qquad (n = 1, 2, \ldots) \tag{2.1}$$

where operator $d^n$ is determined by formula

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z}d^{n+1}(u) = d^n_{zzz} + 4ud^n_z + 2u_z d^n \\ &d^0 = \tfrac{1}{2} \qquad d^1 = u \end{aligned} \tag{2.2}$$

and its hierarchy is in the form [10–13]

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + 2v\right) d^n(v_z - v^2) - zv - \alpha = 0 \qquad (n = 1, 2, \ldots). \tag{2.3}$$

We have the first Painlevé equation

$$u_{zz} + 3u^2 - \frac{z}{2} = 0 \tag{2.4}$$

from equations (2.1) at $n = 1$.

If we take $n = 2$ in equations (2.1) we obtain the fourth-order equation in the form

$$u_{zzzz} + 5u_z^2 + 10uu_{zz} + 10u^3 - \frac{z}{2} = 0. \tag{2.5}$$

By analogy we obtain the second Painlevé equation

$$v_{zz} - 2v^3 - zv - \alpha = 0 \tag{2.6}$$

from equations (2.3) at $n = 1$. If we take $n = 2$ in equations (2.3) we find the fourth-order equation which takes the form

$$v_{zzzz} - 10v^2v_{zz} - 10vv_z^2 + 6v^5 - zv - \alpha = 0. \tag{2.7}$$

It is known that equations (2.4) and (2.6) determine new functions which are the Painlevé transcendents. The question arises as to whether there are new functions determined by equations (2.1) and (2.3) at $n \geqslant 2$.

To answer this question we need in investigation of equations (2.1) and (2.3) on the Painlevé property in the beginning and thereafter we have to show that the general solutions of these equations are the essentially transcendentical functions with respect to their constants of integration.

Recently we studied some properties of equations (2.1) and (2.3) and we now know that these equations possess the Painlevé property because of the following reasons [14].

First, these equations were obtained as reductions of nonlinear partial differential equations which are solved by inverse scattering transform. Taking into account the conjecture of Ablowitz *et al* [8, 9] one expects that equations (2.1) and (2.3) possess the Painlevé property.

Secondly, we checked equations (2.5) and (2.7) with the Painlevé test using the algorithm of Conte *et al* [15]. These equations passed the Painlevé test [14].

Thirdly, we found the Lax pairs for equations (2.1) and (2.3) and we will now be able to solve these equations. It is known [1, 3] that 'good' Lax pair is the sufficiency condition for the integrability of the original equation. As this takes place application of the Gelfand–Levitan–Marchenko integral equation gives the algorithm for the solution of the Cauchy problem and strict proof of the Painlevé property for nonlinear equations. We obtained the Lax pairs for equations (2.1) and (2.3) which are 'good' because they were used for solving equations (2.4) and (2.6) in the partial case.

Taking into account the above-mentioned reasons we suppose that the property of uniformization for the general solutions of equations (2.1) and (2.3) is carried out.

Let us now also consider the partial differential equation which takes the form

$$q_t + \frac{\partial}{\partial x}(q_{xxxx} + 5q_x q_{xx} - 5q^2 q_{xx} - 5qq_x^2 + q^5) = 0. \tag{2.8}$$

This equation was first written by Fordy and Gibbons [16] and can also be solved by the inverse scattering transform. Moreover, equation (2.8) passes the Painlevé test [17, 18].

Equation (2.8) admits the Lie group transformation [19] and therefore has the special solution in the form

$$q(x,t) = (5t)^{-\frac{1}{5}} w(z) \qquad z = x(5t)^{-\frac{1}{5}}. \tag{2.9}$$

As this takes place the equation for $w(z)$ takes the form [20]

$$w_{zzzz} + 5w_z w_{zz} - 5w^2 w_{zz} - 5ww_z^2 + w^5 - zw - \gamma = 0. \tag{2.10}$$

Equation (2.10) may remind us of equation (2.7) but it is clear that they differ. This equation was obtained by the reduction of the integrable equation (2.8) and therefore one can expect that equation (2.10) possesses the Painlevé property as equations (2.5) and (2.7).

It should be noted that there is a map for equation (2.8) which connects this equation with the singular manifold equation [21, 22]. It takes the form

$$\begin{aligned} q_t &+ \frac{\partial}{\partial x}(q_{xxxxx} + 5q_x q_{xx} - 5q^2 q_{xx} - 5qq_x^2 + q^5) \\ &= \frac{\partial}{\partial x}\left(\frac{1}{\varphi_x}\frac{\partial}{\partial x}\right)[\varphi_t + \varphi_x(\{\varphi; x\}_{xx} + 4\{\varphi; x\}^2)] \end{aligned} \tag{2.11}$$

where

$$q = \frac{\varphi_{xx}}{\varphi_x} \tag{2.12}$$

and $\{\varphi; x\}$ is the Schwarzian derivative [23]

$$\{\varphi; x\} = \frac{\varphi_{xxx}}{\varphi_x} - \frac{3}{2}\frac{\varphi_{xx}^2}{\varphi_x^2}. \tag{2.13}$$

Taking into account variables (2.9) and

$$\varphi(x,t) = \varphi(z) \qquad \Psi(x,t) = \Psi(z) \qquad z = x(5t)^{-\frac{1}{5}} \tag{2.14}$$

one can obtain some relation from identity (2.11). This takes the form

$$w_{zzzz} + 5w_z w_{zz} - 5w^2 w_{zz} - 5ww_z^2 + w^5 - zw - \frac{1}{2} = \left(\frac{\mathrm{d}}{\mathrm{d}z} + w\right)\left(F_{zz} + 4F^2 - \frac{z}{2}\right) \qquad (2.15)$$

where

$$F = w_z - \tfrac{1}{2}w^2. \qquad (2.16)$$

From relations (2.15) one can see that there are some special solutions of equation (2.10) at $\gamma = \frac{1}{2}$. These solutions can be obtained by taking into account the solutions of the first Painlevé equations

$$F_{zz} + 4F^2 - \frac{z}{2} = 0 \qquad (2.17)$$

and the Riccatti equations (2.16).

We also have to note that there is some relation for equation (2.7) as (2.15). It takes the form [10]

$$v_{zzzz} - 10v^2 v_{zz} - 10vv_z^2 + 6v^5 - zv - \frac{1}{2} = \left(\frac{\mathrm{d}}{\mathrm{d}z} + 2v\right)\left(w_{zz} + 3w^2 - \frac{z}{2}\right) \qquad (2.18)$$

where

$$v_z - v^2 = w. \qquad (2.19)$$

The relation (2.18) shows that there is a special solution of equation (2.7) which can be found from the Riccatti equation (2.19), taking into account the solution of the first Painlevé equation (2.4).

## 3. Approach applied

The solution of the problem of finding new functions determined by nonlinear ODEs (2.5), (2.7) and (2.10) reduces to the investigation of the functional dependence of their general solutions on the constants of integration. As this takes place three different cases are possible [24–26].

In the first case the general solution of the equation has rational or algebraic dependence on arbitrary constants. This case does not give any new function.

In the second case the general solution of the equation does not have any rational or algebraic dependence on the arbitrary constants but the arbitrary constant can enter the first integral in algebraic form. This case leads to the semi-transcendental function of the general solution with respect to the constants of integration and does not give any new function.

The third case corresponds to the special dependence of the general solution of the equation on constants of integration. This case contradicts the dependence of the general solution of the equation on the constants of integration which were in the first and second cases. They say that this case gives the essentially transcendental function with respect to the constants of integration. Let us note that the six Painlevé transcendents correspond to this case.

Later we wish to show that the general solutions of equations (2.5), (2.7) and (2.10) are the essentially transcendental functions with respect to their constants of integration.

We need to prove that equations (2.5), (2.7) and (2.10) have no first integrals in the polynomial form. To prove this we use the same approach as for the three equations. Let us consider this one.

One can see that equations (2.5), (2.7) and (2.10) can be written in the following form

$$y_{zzzz} + F_i(y, y_z)y_{zz} + G_i(y, y_z, z) = 0 \qquad (i = 1, 2, 3) \qquad (3.1)$$

where

$$F_1 = 10y \qquad G_1 = 5y_z^2 + 10y^3 - \frac{z}{2} \qquad y = u \tag{3.2}$$

$$F_2 = -10y^2 \qquad G_2 = -10yy_z^2 + 6y^5 - zy - \alpha \qquad y = v \tag{3.3}$$

$$F_3 = 5y_z - 5y^2 \qquad G_3 = -5yy_z^2 + y^5 - zy - \gamma \qquad y = w. \tag{3.4}$$

Let us assume that equations (3.1) have the first integrals

$$P_i = P_i(y, y_z, y_{zz}, y_{zzz}, z) = C_i \qquad (i = 1, 2, 3). \tag{3.5}$$

Later we will use the following designations in this section

$$\begin{array}{llll} y_1 = y_z & y_2 = y_{zz} & y_3 = y_{zzz} & y_4 = y_{zzzz} \\ P = P_i & C = C_i, & F = F_i & G = G_i. \end{array} \tag{3.6}$$

Then taking into account the definition of integral (3.5) we obtain the equation in the form

$$E = \frac{\partial P}{\partial z} + \frac{\partial P}{\partial y} y_1 + \frac{\partial P}{\partial y_1} y_2 + \frac{\partial P}{\partial y_2} y_3 + \frac{\partial P}{\partial y_3} y_4 = 0. \tag{3.7}$$

Equation (3.7) on the other hand, has to correspond to equation (3.1) so there is the identity

$$E = Q(y_4 + Fy_2 + G) \tag{3.8}$$

where $Q$ is a polynomial of $y$, $y_1$, $y_2$, $y_3$ and $z$.

One can see from equation (3.8) the equality

$$\frac{\partial P}{\partial y_3} = Q. \tag{3.9}$$

Therefore, equation (3.8) takes the form

$$\frac{\partial P}{\partial z} + \frac{\partial P}{\partial y} y_1 + \frac{\partial P}{\partial y_1} y_2 + \frac{\partial P}{\partial y_2} y_3 - (Fy_2 + G)\frac{\partial P}{\partial y_3} = 0. \tag{3.10}$$

Now let us assume that the first integral of equations (3.1) has the form

$$P = \sum_{k=0}^{m} r_k y_3^{m-k} \tag{3.11}$$

where

$$r_k = r_k(y, y_1, y_2, z). \tag{3.12}$$

Substituting (3.11) into equation (3.10) and equating the same powers of $y_3$ to zero gives the following set of equations

$$\frac{\partial r_0}{\partial y_2} = 0 \tag{3.13}$$

$$\frac{\partial r_1}{\partial y_2} + \frac{\partial r_0}{\partial z} + \frac{\partial r_0}{\partial y} y_1 + \frac{\partial r_0}{\partial y_1} y_2 = 0 \tag{3.14}$$

$$\frac{\partial r_2}{\partial y_2} + \frac{\partial r_1}{\partial z} + \frac{\partial r_1}{\partial y} y_1 + \frac{\partial r_1}{\partial y_1} y_2 = r_0(Fy_2 + G) \tag{3.15}$$

$$\frac{\partial r_{k+1}}{\partial y_2} + \frac{\partial r_k}{\partial z} + \frac{\partial r_k}{\partial y} y_1 + \frac{\partial r_k}{\partial y_1} y_2 = (m - k + 1)r_{k-1}(Fy_2 + G) \qquad (k = 2, \ldots, m-1) \tag{3.16}$$

$$\frac{\partial r_m}{\partial z} + \frac{\partial r_m}{\partial y} y_1 + \frac{\partial r_m}{\partial y_1} y_2 = r_{m-1}(Fy_2 + G) \tag{3.17}$$

One can see that all coefficients $r_k (k = 0, 1, \ldots, m-1)$ can be found from equations (3.13)–(3.16). As this takes the place the solutions $r_{m-1}$ and $r_m$ have to satisfy equation (3.17) if $P$ is a integral of equations (3.1).

We have

$$r_0 = r_0(y, y_1, z) \tag{3.18}$$

from equation (3.13).

The solution of equation (3.14) can be presented in the form

$$r_1 = -\frac{1}{2}\frac{\partial r_0}{\partial y_1} y_2^2 - b_0 y_2 + f_1(y, y_1, z) \tag{3.19}$$

where

$$b_0 = \frac{\partial r_0}{\partial z} + \frac{\partial r_0}{\partial y} y_1. \tag{3.20}$$

Equating the same powers of $y_2$ in equation (3.17) gives

$$\frac{\partial^{m+1} r_0}{\partial y_1^{m+1}} = 0 \qquad \frac{\partial^m b_0}{\partial y_1^m} = 0. \tag{3.21}$$

In fact, one can find the general form of polynomial dependence $r_0$ on $y$, $y_1$ and $z$ by solving equations (3.13)–(3.17) and taking into account (3.21) but one notes that $r_0$ is contained in $r_k$ as a linear expression. Without loss of generality let us take that

$$r_0 = y_1^m. \tag{3.22}$$

Now one can write

$$r_1 = -\tfrac{1}{2} m y_1^{m-1} y_2^2 + f_1(y, y_1, z) \tag{3.23}$$

and the solution of equation (3.15) takes the form

$$r_2 = \frac{1}{8} m(m-1) y_1^{m-2} y_2^4 + \frac{1}{2}\left(F y_1^m - \frac{\partial f_1}{\partial y_1}\right) y_2^2 - \left[G y_1^m - \left(\frac{\partial f_1}{\partial z} + \frac{\partial f_1}{\partial y} y_1\right)\right] y_2 + f_2(y, y_1, z). \tag{3.24}$$

One can assume that

$$r_k = a_k y_2^{2k} + b_k y_2^{2k-2} + c_k y_2^{2k-3} + \cdots \tag{3.25}$$

where the coefficients $a_k$, $b_k$ and $c_k$ depend on $y$, $y_1$ and $z$.

Substituting (3.25) into equation (3.16) and equating the same powers of $y_2$ leads to the following recursion formulae

$$a_{k+1} = -\frac{1}{2k+2}\frac{\partial a_k}{\partial y_1} \tag{3.26}$$

$$b_{k+1} = \frac{1}{2k}\left[(m-k+1)\, a_{k-1} F - \frac{\partial b_k}{\partial y_1}\right] \tag{3.27}$$

$$c_{k+1} = \frac{1}{2k-1}\left[(m-k+1)\, a_{k-1} G - \frac{\partial c_k}{\partial y_1} - \left(\frac{\partial b_k}{\partial z} + y_1 \frac{\partial b_k}{\partial y}\right)\right]. \tag{3.28}$$

We have, on the other hand, from equation (3.17)

$$\frac{\partial a_m}{\partial y_1} = 0 \tag{3.29}$$

$$\frac{\partial b_m}{\partial y_1} = a_{m-1} F \tag{3.30}$$

$$\frac{\partial c_m}{\partial y_1} + \frac{\partial b_m}{\partial z} + y_1 \frac{\partial b_m}{\partial y} = a_{m-1} G. \tag{3.31}$$

Solutions of equations (3.26) and (3.29) take the form

$$a_k = (-1)^k \frac{m(m-1)\dots(m-k+1)}{2^k k!} y_1^{m-k} \tag{3.32}$$

so that

$$a_{m-1} = (-1)^{m-1} \frac{m}{2^{m-1}} y_1 \qquad a_m = (-1)^m \frac{1}{2^m}. \tag{3.33}$$

The formulae (3.33) and the sets of equations (3.27), (3.28) and (3.30), (3.31) will be used in the proof of dependences of the solution on constants of integration for equations (2.5), (2.7) and (2.10).

## 4. Transcendents defined by equation (2.5)

Let us consider the dependence of the general solution of equation (2.5) on the constants of integration. In this section we prove the following theorem.

**Theorem 4.1.** *The general solutions of equation (2.5) are the essentially transcendental functions with respect to their constants of integration.*

**Proof.** The proof falls into two parts. First, we need to prove that the general solution has transcendental dependence on the initial conditions. Secondly, we need to prove that the general solution of equation (2.5) is not a semi-transcendental function. Using the following variables [24–26]

$$u = \lambda^{-2} u' \qquad z = \lambda z' \tag{4.1}$$

where $\lambda$ is some parameter, one can transform equation (2.5) into the following one:

$$u_{zzzz} + 10uu_{zz} + 5u_z^2 + 10u^3 - \frac{\lambda^7}{2} z = 0 \tag{4.2}$$

(the primes of the variables are omitted). It is easy to see that equation (4.2) at $\lambda = 0$ is transformed into the stationary Korteveg-de Vries equation of the fifth order, which takes the form

$$u_{zzzz} + 10uu_{zz} + 5u_z^2 + 10u^3 = 0. \tag{4.3}$$

The solution of equation (4.3) was studied in detail by Drach [27] and by Dubrovin [28]. Dubrovin found that the solution of equation (4.3) can be expressed by the theta function on the Riemann surface [28–30]

$$u = \frac{\mathrm{d}^2}{\mathrm{d}z^2} \ln[\theta(az + z_0)] \tag{4.4}$$

where $\theta(z)$ is the theta function on the Riemann surface, $a$ is the vector of periods of some normalized differential and $z_0$ is the arbitrary two-dimensional vector [28]. Solution (4.4) has transcendental dependence on arbitrary constants [28]. Consequently, the general solution of equation (4.2) at $\lambda \neq 0$ also has transcendental dependence on the arbitrary constants.

However, equation (4.3) has the first integral in the form

$$P_4 = u_z u_{zzz} - \tfrac{1}{2} u_{zz}^2 + 5uu_z^2 + \tfrac{5}{2} u^4 = C_4. \tag{4.5}$$

Consequently, the general solution of equation (4.3) is the semi-transcendental function with respect to constants of integration.

Let us show that equation (2.5) (or equation (4.2) at $\lambda \neq 0$) has no first integrals in the polynomial form. For this purpose we use the set of equations (3.27) and (3.28) which can be written in the form

$$b_{k+1} = \frac{1}{2k}\left[(m-k+1)10ua_{k-1} - \frac{\partial b_k}{\partial u_z}\right] \tag{4.6}$$

$$c_{k+1} = \frac{1}{(2k-1)}\left[(m-k+1)\left(5u_z^2 + 10u^3 - \frac{z}{2}\right)a_{k-1} - \frac{\partial c_k}{\partial u_z} - \left(\frac{\partial b_k}{\partial z} + u_z\frac{\partial b_k}{\partial u}\right)\right]$$
$$(k = 1, \ldots, m-1) \tag{4.7}$$

for equation (2.5).

On the other hand, coefficients $b_m$ and $c_m$ have to satisfy the set of equations which can be obtained from equations (3.30) and (3.31)

$$\frac{\partial b_m}{\partial u_z} = 10ua_{m-1} \tag{4.8}$$

$$\frac{\partial c_m}{\partial u_z} + \frac{\partial b_m}{\partial z} + u_z\frac{\partial b_m}{\partial u} = \left(5u_z^2 + 10u^3 - \frac{z}{2}\right)a_{m-1}. \tag{4.9}$$

One obtains

$$b_m = (-1)^{m-1}\frac{m}{2^{m-1}}[5uu_z^2 + g_1(u,z)] \tag{4.10}$$

from equation (4.8). Substituting (4.10) into equation (4.9) and equating the same powers of $u_z$ to zero gives

$$g_1(u,z) = \tfrac{5}{2}u^4 - \tfrac{1}{2}zu + p_1(z) \tag{4.11}$$

where $p_1(z)$ is a function of $z$.

By the method of mathematical induction one obtains the coefficients $b_k$ in the form

$$b_k = (-1)^{k-1}\frac{m(m-1)\ldots(m-k+1)}{2^{k-1}(k-1)!}u_z^{m-k}\left[5uu_z^2 + \frac{5}{2}u^4 - \frac{1}{2}zu + p_1(z)\right] \tag{4.12}$$

from equation (4.6). We also have

$$b_1 = mu_z^{m-1}[5uu_z^2 + \tfrac{5}{2}u^4 - \tfrac{1}{2}zu + p_1(z)] \tag{4.13}$$

from equation (4.12). Taking into account solution (4.13) one can obtain

$$c_2 = -mu_z^{m-1}\left(\frac{\partial p_1}{\partial z} - \frac{1}{2}u\right) \tag{4.14}$$

from equation (4.7).

Assuming that

$$c_k = (-1)^{k-1}A_k u_z^{m-k+1}\left(\frac{\partial p_1}{\partial z} - \frac{1}{2}u\right) \tag{4.15}$$

(where $A_k$ is some positive constant) it leads by mathematical induction to the solution for $c_{k+1}$ in the form

$$c_{k+1} = (-1)^k A_{k+1}u_z^{m-k}\left(\frac{\partial p_1}{\partial z} - \frac{1}{2}u\right) \tag{4.16}$$

where

$$A_{k+1} = \frac{(m-k+1)}{(2k-1)}\left[A_k + \frac{m(m-1)\ldots(m-k)}{2^{k-1}(k-1)!}\right]. \tag{4.17}$$

Thus we have

$$c_m = (-1)^{m-1} A_m u_z \left( \frac{\partial p_1}{\partial z} - \frac{1}{2} u \right). \tag{4.18}$$

Substituting (3.33), (4.12) and (4.18) into equation (4.9) gives the contradiction

$$\left( \frac{m}{2^{m-1}} + A_m \right) \left( \frac{\partial p_1}{\partial z} - \frac{1}{2} u \right) \neq 0. \tag{4.19}$$

This contradiction shows that the integral of equation (2.5) in the form (3.11) does not exist. Consequently the general solutions of equation (2.5) are essentially transcendental functions with respect to their constants of integration. This proves theorem 4.1. □

## 5. Transcendents defined by equation (2.7)

Let us consider the dependence of the general solution of equation (2.7) on the constants of integration. We wish to prove the following theorem.

**Theorem 5.1.** *The general solution of equation (2.7) are essentially transcendental functions with respect to their constants of integration.*

**Proof.** This proof also contains two parts. First, one uses the variables

$$v = \mu^{-1} v' \qquad z = \mu z' \tag{5.1}$$

(where $\mu$ is some parameter) so that equation (2.7) is transformed to the following equation

$$v_{zzzz} - 10 v v_{zz} - 10 v v_z^2 + 6 v^5 - \mu^3 z v - \mu^5 \alpha = 0 \tag{5.2}$$

(the primes of the variables are omitted).

Assuming $\mu = 0$ in equation (5.2) we obtain the equation

$$v_{zzzz} - 10 v v_{zz} - 10 v v_z^2 + 6 v^5 = 0 \tag{5.3}$$

which corresponds to the stationary modified Korteveg-de Vries equation of the fifth order. The general solution of equation (5.3) can be presented via the theta function on the Riemann surface as the solution of equation (4.3) [31]. This solution has transcendental dependence on the constants of integration. Consequently, the general solution of equation (5.2) at $\mu \neq 0$ also has transcendental dependence on the constants of integration. Thus equation (2.7) does not belong to the first case of dependence on the constants.

However, equation (5.3) has the first integral in the form

$$P_5 = v_z v_{zzz} - \tfrac{1}{2} v_{zz}^2 - 10 v^2 v_z^2 + v^6 = C_5. \tag{5.4}$$

Consequently equation (5.3) has the general solution in the form of the semi-transcendental function with respect to the initial conditions. We show this is not the case for equation (2.7).

For this we again use the set of equations (3.27), (3.28) which can be presented in the following form

$$b_{k+1} = \frac{1}{2k} \left[ -10 v^2 (m - k + 1) a_{k-1} - \frac{\partial b_k}{\partial v_z} \right] \tag{5.5}$$

$$c_{k+1} = \frac{1}{(2k-1)} \left[ (-10 v v_z^2 + 6 v^5 - z v - \beta)(m - k + 1) a_{k-1} - \frac{\partial c_k}{\partial v_z} - \left( \frac{\partial b_k}{\partial z} + u_z \frac{\partial b_k}{\partial v} \right) \right]$$

$$(k = 1, \ldots, m - 1) \tag{5.6}$$

for equation (2.7).

On the other hand, the coefficients $b_m$ and $c_m$ have to satisfy the set of equations

$$\frac{\partial b_m}{\partial v_z} + 10v^2 a_{m-1} = 0 \tag{5.7}$$

$$\frac{\partial c_m}{\partial v_z} + \frac{\partial b_m}{\partial z} + v_z \frac{\partial b_m}{\partial v} + (10vv_z^2 - 6v^5 + zv + \beta)a_{m-1} = 0 \tag{5.8}$$

which is obtained from equations (3.30) and (3.31).

We have

$$b_m = (-1)^m \frac{m}{2^{m-1}}[-5v^2 v_z^2 + g_2(v, z)] \tag{5.9}$$

from equation (5.7).

Substituting solution (5.9) into equation (5.8) and equating the same powers of $v_z$ to zero gives

$$g_2(u, z) = v^6 - \tfrac{1}{2}zv^2 - \beta v + p_2(z) \tag{5.10}$$

where $p_2(z)$ is a function of integration.

By the method of mathematical induction we obtain

$$b_k = (-1)^{k+1} \frac{m(m-1)\ldots(m-k+1)}{2^{k-1}(k-1)!} v_z^{m-k} \left[ v^6 - 5v^2 v_z^2 - \frac{1}{2}zv^2 - \beta v + p_2(z) \right] \tag{5.11}$$

and

$$b_1 = mv_z^{m-1}[v^6 - 5v^2 v_z^2 - \tfrac{1}{2}zv^2 - \beta v + p_2(z)]. \tag{5.12}$$

Now one can find the coefficient $c_2$ from equation (5.6). It takes the form

$$c_2 = -mv_z^{m-1} \left( \frac{\partial p_2}{\partial z} - \frac{1}{2}v^2 \right). \tag{5.13}$$

Taking into account the method of mathematical induction we obtain

$$c_{k+1} = (-1)^k A_{k+1} v_z^{m-k} \left( \frac{\partial p_2}{\partial z} - \frac{1}{2}v^2 \right) \tag{5.14}$$

where $A_{k+1}$ can be expressed by the recursion formula (4.17).

We have

$$c_m = (-1)^{m-1} A_m v_z \left( \frac{\partial p_2}{\partial z} - \frac{1}{2}v^2 \right) \tag{5.15}$$

from equation (5.14).

Substituting (3.33), (5.9) and (5.15) into equation (5.8) gives the contradiction

$$\left( \frac{m}{2^{m-1}} + A_m \right) \left( \frac{\partial p_2}{\partial z} - \frac{1}{2}v^2 \right) \neq 0. \tag{5.16}$$

The integral of equation (2.7) in the form (3.11) does not, therefore, exist. Consequently the general solution of equation (2.7) are the essentially transcendental functions with respect to their constants of integration. □

## 6. Transcendents defined by equation (2.10)

By way of the last example let us consider equation (2.10). Using the variables

$$w = \eta^{-1} w' \qquad z = \eta z \tag{6.1}$$

(where $\eta$ is some parameter) one can transform equation (2.10) to the following one

$$w_{zzzz} + 5w_z w_{zz} - 5w^2 w_{zz} - 5ww_z^2 + w^5 - \eta^3 zw - \eta^5 \gamma = 0 \tag{6.2}$$

(the primes in the variables are also omitted).

Equation (6.2) at $\eta = 0$ takes the form of the stationary Fordy–Gibbons equation (2.8)

$$w_{zzzz} + 5w_z w_{zz} - 5w^2 w_{zz} - 5ww_z^2 + w^5 = 0. \tag{6.3}$$

The general solution of this equation can also be expressed via the theta function on the Riemann surface [31]. This solution has transcendental dependence on the initial conditions. Consequently, one can expect that the general solution of equation (6.2) has transcendental dependence on the constants of integration at $\eta \neq 0$ as well. However, equation (6.3) has the first integral in the form

$$P_6 = w_z w_{zzz} + \tfrac{5}{3} w_z^3 - \tfrac{5}{2} w^2 w_z^2 + \tfrac{1}{6} w^6 - \tfrac{1}{2} w_{zz}^2 = C_6 \tag{6.4}$$

and consequently the solution of equation (6.3) is the semi-transcendental function with respect to their initial condition.

Let us show that equation (2.10) does not have any integral in the form (3.11). For this purpose we again use the set of equations (3.27), (3.28). These ones can be written in the form

$$b_{k+1} = \frac{1}{2k}\left[(m-k+1)(5w_z - 5w^2)a_{k-1} - \frac{\partial b_k}{\partial w_z}\right] \tag{6.5}$$

$$c_{k+1} = \frac{1}{(2k-1)}\left[(m-k+1)(w^5 - 5ww_z^2 - zw - \gamma)a_{k-1} - \frac{\partial c_k}{\partial w_z} - \left(\frac{\partial b_k}{\partial z} + w_z \frac{\partial b_k}{\partial w}\right)\right]$$
$$(k = 1, \ldots, m-1). \tag{6.6}$$

Moreover, the coefficients $b_m$ and $c_m$ have to satisfy the set of equations

$$\frac{\partial b_m}{\partial w_z} = (5w_z - 5w^2)a_{m-1} \tag{6.7}$$

$$\frac{\partial c_m}{\partial w_z} + \frac{\partial b_m}{\partial z} + w_z \frac{\partial b_m}{\partial w} = (w^5 - 5ww_z^2 - zw - \gamma)a_{m-1}. \tag{6.8}$$

Solution of equation (6.7) can be presented in the form

$$b_m = (-1)^{m-1}\frac{m}{2^{m-1}}\left[\frac{5}{3}w_z^3 - \frac{5}{2}w^2 w_z^2 + g_3(w, z)\right]. \tag{6.9}$$

Substituting (6.9) into equation (6.8) and equating of the same powers of $w_z$ to zero leads to the solution for $g_3(w, z)$ in the form

$$g_3(w, z) = \tfrac{1}{6}w^6 - \tfrac{1}{2}zw^2 - \gamma w + p_3(z). \tag{6.10}$$

By the method of mathematical induction we obtain

$$b_k = (-1)^{k-1}\frac{m(m-1)\ldots(m-k+1)}{2^{k-1}(k-1)!}w_z^{m-k}$$
$$\times\left[\frac{5}{3}w_z^2 - \frac{5}{2}^2 w_z^2 + \frac{1}{6}w^6 - \frac{1}{2}zw^2 - \gamma w + p_3(z)\right] \qquad (k = 1, \ldots, m-1). \tag{6.11}$$

One can find $c_2$ in the form

$$c_2 = -m w_z^{m-1} \left( \frac{\partial p_3}{\partial z} - \frac{1}{2} w^2 \right) \tag{6.12}$$

from equation (6.6).

By again taking into account the method of the mathematical induction we have

$$c_{k+1} = (-1)^k A_{k+1} w_z^{m-1} \left( \frac{\partial p_3}{\partial z} - \frac{1}{2} w^2 \right) \tag{6.13}$$

where $A_{k+1}$ can be also expressed by the recursion formula (4.17).

Using

$$c_m = (-1)^{m-1} A_m w_z \left( \frac{\partial p_3}{\partial z} - \frac{1}{2} w^2 \right) \tag{6.14}$$

and (3.33) and (6.9) we have

$$\left( \frac{m}{2^{m-1}} + A_m \right) \left( \frac{\partial p_3}{\partial z} - \frac{1}{2} w^2 \right) \neq 0 \tag{6.15}$$

from equation (6.8).

We showed that the integral of equation (2.10) in the form (3.11) does not exist. This shows that the general solution of equation (2.10) is the transcendental function on the initial conditions.

## 7. On irreducibility of equations (2.5), (2.7) and (2.10)

We proved in sections 4–6 that equations (2.5), (2.7) and (2.10) do not have any first integrals in the polynomial form and that their solutions are the essentially transcendental functions with respect to the constants of integration. We will discuss the problem of irreducibility of these equations in this section. It is known that the notions of irreducibility and the transcendental dependence on initial conditions are equivalent for the second algebraic differential equations [2], but there is not such proof of equivalence at higher-order equations. The problem of finding new transcendents defined by nonlinear ODEs (2.5), (2.7) and (2.10) is only one in the investigation of dependence on the solutions of the Painlevé equations.

Actually, one can see from relations (2.15) and (2.18) that the special solutions of equations (2.7) and (2.10) are expressed via the solutions of the first Painlevé equations and one can imagine that solutions of equations (2.5), (2.7) and (2.10) are expressed via the solutions of the Painlevé equations in the general case.

It is known, for example, that the third-order equation which takes the form

$$y_3 + 6 y y_1 - z y_1 - 2y = 0 \tag{7.1}$$

where

$$y_1 = \frac{\mathrm{d}y}{\mathrm{d}z} \qquad y_3 = \frac{\mathrm{d}^3 y}{\mathrm{d}z^3}.$$

The solution of this equation is the transcendental function with respect to constants of integration but this one is not the new transcendent because the solution of equation (7.1) is expressed by the formula

$$y = v_1 - v^2 \tag{7.2}$$

where $v$ is the solution of the second Painlevé equation (2.6). The matter is that equation (7.1) is reducible one.

However, regarding equations (2.5), (2.7) and (2.10) this is not the case because there are reasons to believe that equations (2.5), (2.7) and (2.10) are irreducible.

For example, let us assume that the solution of equation (2.5) is expressed via the solution of the second Painlevé equation corresponding to the following transformation

$$y = g(v, v_1, z) \tag{7.3}$$

where $y$ is a solution of equation (2.5) and $v$ is a solution of equation (2.6). As this takes place one notes that we do not need to take the transformation (7.3) in more general form because it can be transformed to the transformation (7.3) taking into account equation (2.6).

It is easy to check that the transformation (7.3) cannot be found for equation (2.5) and (2.6).

One can suggest closely approximating arguments for transformations (7.3) which connect equations (2.5), (2.7) and (2.10) with other Painlevé equations.

Now let us assume that there exists an expression

$$W = D(y, y_1, y_2, z) \tag{7.4}$$

in equation (2.5) so that $W$ satisfies the first Painlevé equation.

Taking into account (7.4) one can obtain

$$W_x = D_x + D_y y_1 + D_{y_1} y_2 + D_{y_2} y_3 \tag{7.5}$$

$$\begin{aligned} W_{xx} = {} & D_{xx} + D_{yx} y_1 + D_y y_2 + D_{y_1 x} y_2 + D_{y_1} y_3 + D_{y_2 x} y_3 + D_{y_2} y_4 + D_{xy} y_1 + D_{yy} y_1^2 \\ & + D_{y_1 y} y_2 y_1 + D_{y_2 y} y_3 y_1 + D_{x y_1} y_2 + D_{y y_1} y_1 y_2 + D_{y_1 y_1} y_2^2 + D_{y_2 y_1} y_3 y_2 \\ & + D_{x y_2} y_3 + D_{y y_2} y_1 y_3 + D_{y_1 y_2} y_2 y_3 + D_{y_2 y_2} y_3^2. \end{aligned} \tag{7.6}$$

Substituting (7.4) and (7.6) into equation (2.4) we do not obtain equation (2.5). This contradiction shows that we cannot find the expression (7.4) in equation (2.5) which reduces equation (2.5) to (2.4).

Finally, one can suggest more arguments why one can expect that the solutions of equations (2.5), (2.7) and (2.10) are new transcendents.

Let us take the variables

$$u = z^{\frac{1}{3}} \omega \qquad x = \tfrac{6}{7} z^{\frac{7}{6}} \tag{7.7}$$

then equation (2.5) can be presented in the form

$$\begin{aligned} & \omega_{xxxx} + 5\omega_x^2 + 10\omega\omega_{xx} + 10\omega^3 - \frac{\lambda^7}{2} + \frac{2}{x}\omega_{xxx} + \frac{10}{x}\omega\omega_x - \frac{41}{49}\frac{1}{x^2}\omega_{xx} \\ & \qquad - \frac{60}{49}\frac{1}{x^2}\omega^2 + \frac{41}{49}\frac{1}{x^3}\omega_x - \frac{1280}{2401}\frac{1}{x^4}\omega = 0. \end{aligned} \tag{7.8}$$

One can see from equation (7.8) that this equation takes the form

$$\omega_{xxxx} + 5\omega_x^2 + 10\omega\omega_{xx} + 10\omega^3 - \frac{\lambda^7}{2} = 0 \tag{7.9}$$

at $|x| \to \infty$. The solution of equation (7.9) is expressed in terms of the theta function on the Riemann surface The asymptotic solution of equation (2.5) takes the form

$$u(z) \sim z^{\frac{1}{3}} \omega(\tfrac{6}{7} z^{\frac{7}{6}}) \tag{7.10}$$

where $\omega(z)$ is a solution of equation (7.9). The asymptotic solution of equation (2.5) corresponds to the solution of the irreducible equation and consequently one can expect that equation (2.5) is also the irreducible equation. Certainly the rigorous proof of the irreducibility of equations (2.5), (2.7) and (2.10) is derivable from group theory but the above-mentioned arguments allow us to expect that equations (2.5), (2.7) and (2.10) give new transcendents.

## 8. Conclusion

Thus we have shown that the general solutions of equations (2.5), (2.7) and (2.10) are essentially transcendentical functions with respect to their constants of integration.

Actually, these equations possess the Painlevé property and their general solutions are the single-valued functions. In the approximate limit solutions of these equations can be presented via the theta functions on the Riemann surfaces which are the semi-transcendentical functions with respect to their constants of integration. Consequently, these solutions have transcendental dependences on their constants.

We have shown that equations (2.5), (2.7) and (2.10) have no first integrals in the polynomial form. Consequently, their general solutions are the essentially transcendental functions with respect to their constants of integration. These solutions belong to the class of functions as the six Painlevé transcendents.

We have discussed the irreducibility of equations (2.5), (2.7) and (2.10) and have presented some reasons why one can expect that these equations are irreducible ones. We believe that these equations can give new transcendents defined by nonlinear ODEs.

In fact, we think that every general solution of equations (2.1) and (2.3) are the transcendents defined by nonlinear ODE and therefore we hope to obtain the infinite number of such transcendents.

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